A comparison of stock return distribution models

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1 Introduction

Geometric Brownian motion (GBM) is the standard modeling method for stock return distributions; however, emerging work has explored the application of the quantum harmonic oscillator (QHO) for the purpose of capturing local oscillations due to restorative market forces [1].

The restorative dynamics that make the quantum harmonic oscillator attractive for this application may also be captured by modifying the traditional GBM equations by combining features of the Ornstein-Uhlenbeck Process and geometric Brownian motion. The Ornstein-Uhlenbeck Process is a mean reverting process, demonstrating a tendency to drift towards its long-term mean. Like the restorative force modeled by the QHO, the mean restorative attraction in this modified model is greater at larger distances from the mean. This approach may combine attractive features of both the standard GBM and the QHO.

For the purpose of this exploration, we examined the ability of GBM, a modified GBM with an Ornstein-Uhlenbeck mean reversion term, and the QHO to tmodel the behaviors of a return distribution over time. The models were applied to S&P 500 returns over the last five years. For each model, parameters were chosen by minimizing a goodness of fit statistic using parameter search algorithms. The optimal results for each model were compared, using the Cramér von Mises test statistic for multiple return windows.

2 Model

2.1 Formulation

The equations for geometric Brownian motion (1) and the Ornstein-Uhlenbeck Process (2) are well known.

$$dX_t = \alpha X_t dt + \sigma X_t dB_t \tag{1}$$

$$dX_t = \alpha(\mu - X_t)dt + \sigma dB_t \tag{2}$$

These two equations were modified, retaining the μ term in the Ornstein-Uhlenbeck process and the respective σ terms of each equation. The resulting equation is given in (3).

$$dX_t = \alpha(\mu(t) - X_t)dt + \sigma_1 X_t dB_t + \sigma_2 dB_t \tag{3}$$

The time-independent Schrödinger equation (4), is also well known. Prior work by Ahn and colleagues [1], derived the appropriate Fokker-Plank equation for the probability density, where H_n is the appropriate Hermite polynomial and E_n is the energy level of the nth eigenstate (5).

$$[\frac{-\hbar^2}{2\mu}\nabla^2 + V(r)]\psi(r) = E\psi(r) \tag{4}$$

$$\rho(x,t) = \sum_{n=0}^{\inf} \frac{A_n}{\sqrt{2^n n!}} \sqrt{\frac{m\omega}{\pi\hbar}} exp(-E_n t) H_n(\sqrt{\frac{m\omega}{\hbar}} x) exp(-\frac{m\omega}{\hbar} x^2) \tag{5}$$

For the purpose of fitting the simulation, equation (5) is simplified to equation (6), where ρ_n is given in equation (7).

$$\rho(x,t) = \sum_{n}^{\inf} C_n(t)\rho_n(x) \tag{6}$$

$$\rho_n(x) = H_n(\sqrt{\frac{m\omega}{\hbar}} x) exp(-\frac{m\omega}{\hbar} x^2) \tag{7}$$

2.2 Solving the modified stochastic differential equation

The combined geometric Brownian motion and the Ornstein-Uhlenbeck Process has the form of the general linear stochastic differential equation, and may be solved likewise [2].

The equation (8) has solution (9), where the first integral is a Riemann integral and the second is an Itó integral.

$$dX_t = \alpha(\mu(t) - X_t)dt + \sigma_1 X_t dB_t + \sigma_2 dB_t \tag{8}$$

$$X_t = X_0 + \int_0^t (\alpha\mu(t) - \alpha X_t)dt + \int_0^t (\sigma_1 + \sigma_2 X_t)dB_t \tag{9}$$

This solution as the form of (10), in which the general solution (11) holds where Y is the solution to the homogeneous equation given in (12).

$$X_t = X_0 + \int_0^t [c_1(s)X_s + c_s(s)]ds + \int_0^t [\sigma_1(s)X_s + \sigma_2(s)]dB_s \ t \in [0, T] \tag{10}$$

$$X_t = Y_t(X_0 + \int_0^t [c_2(t) - \sigma_1(t)\sigma_2(t)]Y_s^{-1}ds + \int_0^t \sigma_2(s)Y_s^{-1}dB_s), \ t \in [0, T] \tag{11}$$

$$Y_t = X_0 \exp(\int_0^t [c_1(s) - 0.5\sigma_1^2(s)]ds + \int_0^t \sigma_1(s)dB_s, t \in [0, T] \tag{12}$$

In terms of the our original equation, our solution is:

$$X_t = Y_t(X_0 + \int_0^t [-\alpha - \sigma_1\sigma_2]Y_s^{-1}ds + \int_0^t \sigma_2 Y_s^{-1}dB_s), \ t \in [0, T] \tag{13}$$

$$Y_t = X_0 \exp(\int_0^t \alpha\mu(s)ds - 0.5\sigma_1^2 t + \sigma_1 B_t \ , t \in [0, T] \tag{14}$$

Rather than using the long term average as μ, here μ is given as a function of time in order to describe the local reversion behavior of the model. In the simulation, this will be captured using an additional parameter γ, which is the window of prior data to be used for mean calculation in the simulation. Alpha is constrained to be greater than zero.

3 Data

Returns for the S&P 500 last five years were calculated for each $\tau \in [1, 20]$, using equation (15). Summary statistics for the resulting data sets are given in Table 1.

$$R_t^{ann} = \frac{252.5}{\tau} ln(\frac{S_{t+\tau}}{S_t}) \tag{15}$$

Table 1: Summary of stock returns for different holding periods (τ)

τ	No. of obs.	Mean	Std.	Skewness	Excess kurtosis
1	2769	0.06	3.19	-0.36	10.94
5	2765	0.06	1.27	-1.1	8.88
20	2750	0.06	0.59	-1.62	7.56

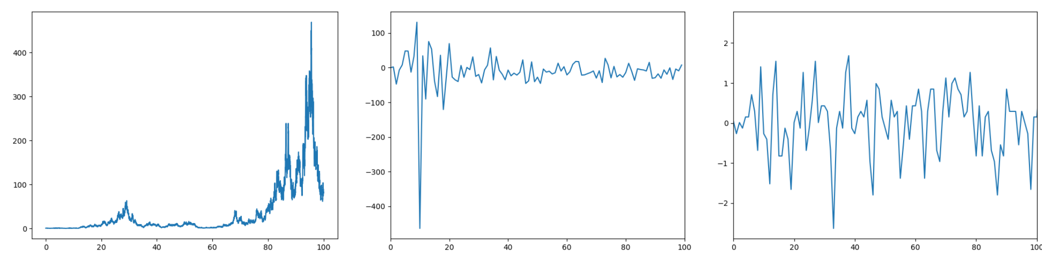
4 Results

We performed our optimization using both a particle swarm and simulated annealing algorithm, selecting the lower of the two results. The optimization algorithm was performed against to minimize the Cramér von Mises test statistic in (16) (where Θ are the parameter values) [3]. In this calculation, $r_j = R_j - \bar{R}$, where R_j is the jth ordered return and $\bar{R}$ is the empirical historical average of the S&P 500 returns. For the QHO, we fitted the first 5 coefficients and the $m\omega$ term. The GBM fitted the μ and σ terms. The modified GBM fitted for the expanded parameters set, α, σ_1, σ_2, and number of prior steps to include in the mean calculation, μ_{steps}. Resulting parameter values are given in Table 2. Simulated trajectories for each model using the $\tau = 20$ results are shown in Figure 1.

$$T(\Theta) = \frac{1}{12M} + \sum_{j=1}^{M} [F(r_j; \Theta) - \frac{j - 1/2}{M}]^2 \tag{16}$$

Table 2: Parameter search results

Models	Parameters	$\tau = 1$	$\tau = 20$
GBM	μ	-3.035	1.563
	σ	0.481	0.119
	T	239.404	151.625
Modified GBM	α	0	0.157
	σ_1	-0.2	-0.2
	σ_2	-0.2	-0.115
	μ_{steps}	20	50
	T	998.620	877.466
QHO	C_0	0	0.2
	C_1	0	0.2
	C_2	0.004	0.086
	C_3	0.053	0.182
	C_4	0.061	0.133
	$m\omega$	1	0.928
	T	296.607	552.782



(a) GBM (b) Modified GBM (c) QHO

Figure 1: Simulated trajectories for each model with $\tau = 20$.

5 Discussion and Conclusion

Overall, the standard GBM model had the lowest Cramér von Mises test statistic across values of τ. The QHO had the second best performance, and the modified GBM performed the worst. This, however, is inadequate to preclude the utility of the modified GBM for modeling purposes. The optimization algorithms were run for fixed time; however, the computations for both the QHO and the modified GBM were significantly more intensive than the standard GBM. In particular, the computation of the stochastic integral was longer than all other processes. This resulted in less iterations of particle adjustment in the particle swarm algorithm and anneal steps in the simulated annealing. Also, some of the parameter search results turned the edge cases, suggesting that expanded parameter ranges may have yielded lower error combinations.

From a qualitative standpoint, Figure 1 demonstrates the similarities between the QHO and the modified GBM. The modifications of the Ornstein-Uhlenbeck process and geometric Brownian motion successfully capture the target oscillatory behaviors. This suggests that there may be further grounds for exploration, with a more rigorous parameter search process.

6 References

[1] Ahn, K., Choi, M.Y., Dai, B., Sohn, S., and Yang, B. Modeling stock return distributions with a quantum harmonic oscillator.

[2] Mikoshm T. Elementary stochastic calculus with finance in view, Vol 6. 2000.

[3] Ataullah, A., Davidson, I. and Tippett, M., A wave function for stock market returns. Physica A: Statistical Mechanics and its Applications, 2009, 388(4), 455-461.